# 4.2

# Algebra of Complex Numbers

## Learning Objectives:

- To define addition, subtraction, multiplication and division of complex numbers and to study their properties
  AND
- To practice the related problems

## Addition and Subtraction

The addition of two complex numbers, $z_1$ and $z_2$, in general gives another complex number. The real components and the imaginary components are added separately in a manner similar to the familiar addition of real numbers.

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) \end{aligned}$$

For example,

$$(2 + 3i) + (4 - 5i) = (2 + 4) + (3 - 5)i = 6 - 2i,$$

obtained by adding the real parts and the imaginary parts.

By straightforward application of the commutativity and associativity of the real and imaginary parts separately, we can show that the addition of complex numbers is commutative and associative.

$$z_1 + z_2 = z_2 + z_1$$

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$$

Thus it is immaterial in what order complex numbers are added.

**Example**: Sum the complex numbers $1 + 2i, 3 - 4i, -2 + i$

**Solution** Summing the real terms we obtain

$$1 + 3 - 2 = 2$$

And summing the imaginary terms we obtain

$$2i - 4i + i = -i$$

Hence

$$(1 + 2i) + (3 - 4i) + (-2 + i) = 2 - i$$

The subtraction of complex numbers is very similar to their addition. In the subtraction of one complex number from the other, we subtract the real parts and subtract the imaginary parts. For example,

$$(2 + 3i) - (4 - 5i) = (2 - 4) + (3 + 5)i = -2 + 8i$$

As in the case of real numbers, if two identical complex numbers are subtracted then the result is zero.

## Multiplication

Complex numbers may be multiplied together and in general give a complex number as the result. The product of two complex numbers $z_1$ and $z_2$, is found by multiplying them out in full and using $i^2 = -1$.

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \end{aligned}$$

In the multiplication of two complex numbers, we carry out the multiplication as if the numbers were ordinary binomials and replace $i^2$ by $-1$.

$$(2+3i)(4-5i) = 8 + 2i - 15i^2 = 8 + 2i - 15(-1)$$
$$= 23 + 2i$$

The multiplication of complex numbers is both commutative and associative.

$$z_1 z_2 = z_2 z_1$$
$$(z_1 z_2) z_3 = z_1 (z_2 z_3)$$

## Division

The division of two complex numbers $z_1$ and $z_2$, may be written as a quotient in the component form

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \quad , z_2 \neq 0$$

In order to separate the real and imaginary components of the quotient, we multiply both numerator and denominator by the complex conjugate of the denominator.

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}$$
$$= \frac{(x_1 x_2 + y_1 y_2) + i(-x_1 y_2 + x_2 y_1)}{{x_2}^2 + {y_2}^2}$$
$$= \frac{x_1 x_2 + y_1 y_2}{{x_2}^2 + {y_2}^2} + i \frac{-x_1 y_2 + x_2 y_1}{{x_2}^2 + {y_2}^2}$$

To divide two complex numbers, multiply both numerator and denominator of the fraction by the conjugate of the denominator. As an example:

$$\frac{2+3i}{4-5i} = \frac{(2+3i)(4+5i)}{(4-5i)(4+5i)} = \frac{8+22i+15i^2}{16+25} = -\frac{7}{41} + \frac{22}{41} i$$

**If $z \epsilon C$, then $Re(z) = \frac{z+\bar{z}}{2}$ , $Im(z) = \frac{z-\bar{z}}{2i}$**

We have, $z = x + iy \Rightarrow \bar{z} = x - iy$

Therefore, $z + \bar{z} = 2x = 2Re(z)$

$$z - \bar{z} = 2iy = 2iIm(z)$$

Thus, $Re(z) = \frac{z+\bar{z}}{2}$ , $Im(z) = \frac{z-\bar{z}}{2i}$

**The complex conjugate of the sum (difference) of two complex numbers is equal to the sum (difference) of their complex conjugates. *i.e.*,**

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$
$$\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$$

Similarly, **the complex conjugate of the product (quotient) of two complex numbers is equal to the product (quotient) of their complex conjugates. *i.e.*,**

$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$$
$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

**Example**: Show that the complex number $2 + i$ and its conjugate are the roots of the equation $x^2 - 4x + 5 = 0$

**Solution** For $x = 2 + i$, we have

$$(2 + i)^2 - 4(2 + i) + 5 = 4 + i^2 + 4i - 8 - 4i + 5$$

$$= 4 - 1 - 8 + 5 = 0$$

Therefore the complex number is a solution of the equation $x^2 - 4x + 5 = 0$.

The conjugate of the complex number $2 + i$ is $2 - i$.

For $x = 2 - i$, we have

$$(2 - i)^2 - 4(2 - i) + 5 = 4 + i^2 - 4i - 8 + 4i + 5$$

$$= 4 - 1 - 8 + 5 = 0$$

Thus, the complex conjugate $2 - i$ of the complex number $2 + i$ is also a root of the equation $x^2 - 4x + 5 = 0$.

**P1:**

If sum of the complex numbers $\frac{1}{3} - \frac{1}{2}i$ and $\frac{1}{2} + \frac{1}{3}i$ is $5x + iy$, then find the values of $x$ and $y$.

**Solution:**

Given, $\left(\frac{1}{3}-\frac{1}{2}i\right)+\left(\frac{1}{2}+\frac{1}{3}i\right)=5x+iy$

$$\Rightarrow \left(\frac{1}{3}+\frac{1}{2}\right)+i\left(\frac{1}{3}-\frac{1}{2}\right)=5x+iy$$

$$\Rightarrow \qquad \frac{5}{6}+i\left(\frac{-1}{6}\right)=5x+iy$$

Comparing real and the imaginary parts, we get

$$5x=\frac{5}{6} \text{ and } y=\frac{-1}{6}$$

$$\Rightarrow x=\frac{1}{6} \text{ and } y=\frac{-1}{6}$$

**P2:**

If $z_1 = 7 - xi$ , $z_2 = -3 + 5i$ , $z_3 = 2y - 11i$ and $z_1 - z_2 = z_3$, then find $x$ and $y$.

**Solution:**

We have, $z_1 = 7 - xi,\ z_2 = -3 + 5i,\ z_3 = 2y - 11i,$

and $z_1 - z_2 = z_3$

$$\Rightarrow (7 - xi) - (-3 + 5i) = 2y - 11i$$

$$\Rightarrow \quad (7 + 3) - i(x + 5) = 2y - 11i$$

$$\Rightarrow \quad 10 - i(x + 5) = 2y - 11i$$

Comparing real and imaginary parts, we get

$$2y = 10 \quad \text{and} \quad -(x + 5) = -11$$

$$\Rightarrow y = 5 \quad \text{and} \quad x + 5 = 11$$

$$\Rightarrow y = 5 \quad \text{and} \quad x = 6$$

$$\therefore\ x = 6 \ \text{ and } \ y = 5$$

**P3:**

If the product of $(2b+3)+i$ and $1-i(a-2)$ is $2i$, then find the values of $a$ and $b$.

**Solution:**

We have, $[(2b+3)+i][1-i(a-2)]=2i$

$\Rightarrow (2b+3)-i(a-2)(2b+3)+i-i^2(a-2)=2i$

$\Rightarrow (2b+3)-i(a-2)(2b+3)+i+(a-2)=2i$

$\Rightarrow (2b+3+a-2)+i\big(1-(a-2)(2b+3)\big)=2i$

$\Rightarrow (a+2b+1)+i\big(1-(a-2)(2b+3)\big)=2i$

Comparing real and imaginary parts, we get

$a+2b+1=0 \Rightarrow a=-2b-1 \quad ------(1)$

and $1-(a-2)(2b+3)=2 \quad ------(2)$

From (1) and (2), we get

$1-(-2b-1-2)(2b+3)=2$

$\Rightarrow b^2+3b+2=0$

$\Rightarrow (b+1)(b+2)=0$

$\Rightarrow b=-1$ or $b=-2$

If $b=-1, a=-2(-1)-1=1 \Rightarrow a=1, b=-1$

If $b=-2, a=-2(-2)-1=3 \Rightarrow a=3, b=-2$

**P4:**

Evaluate $\frac{2+5i}{3-2i}+\frac{2-5i}{3+2i}$

**Solution:**

$$\frac{2+5i}{3-2i}+\frac{2-5i}{3+2i}=\frac{2+5i}{3-2i}\times\frac{3+2i}{3+2i}+\frac{2-5i}{3+2i}\times\frac{3-2i}{3-2i}$$

$$=\frac{6+19i+10i^2}{9-4i^2}+\frac{6-19i+10i^2}{9-4i^2}$$

$$=\frac{6+19i-10}{9+4}+\frac{6-19i-10}{9+4}=\frac{-8}{13}$$

## IP1:

Find the sum of the complex numbers $-\frac{1}{5}+i$, $1-\frac{1}{5}i$ and $\frac{1}{7}-\frac{1}{3}i$

**Solution:**

We have, $-\frac{1}{5}+i$, $1-\frac{1}{5}i$ and $\frac{1}{7}-\frac{1}{3}i$,

$$\left(-\frac{1}{5}+i\right)+\left(1-\frac{1}{5}i\right)+\left(\frac{1}{7}-\frac{1}{3}i\right)$$

$$=\left(-\frac{1}{5}+1+\frac{1}{7}\right)+i\left(1-\frac{1}{5}-\frac{1}{3}\right)$$

$$=\left(\frac{-7+35+5}{35}\right)+i\left(\frac{15-3-5}{15}\right)$$

$$=\frac{33}{35}+\frac{7}{15}i$$

**IP2:**

If $z_1 = 3 + 5i$ , $z_2 = \frac{1}{5} - \frac{1}{3}i$ , $z_3 = \frac{1}{3} - \frac{1}{5}i$ and $z_4 = 5 + 3i$, then find $(z_1 + z_3) - (z_2 + z_4)$.

**Solution:**

We have, $z_1 = 3 + 5i$ , $z_2 = \frac{1}{5} - \frac{1}{3}i$ , $z_3 = \frac{1}{3} - \frac{1}{5}i$ ,

and $z_4 = 5 + 3i$

$$(z_1 + z_3) - (z_2 + z_4)$$

$$= \left(3 + 5i + \frac{1}{3} - \frac{1}{5}i\right) - \left(\frac{1}{5} - \frac{1}{3}i + 5 + 3i\right)$$

$$= \left(\frac{10}{3} + \frac{24}{5}i\right) - \left(\frac{26}{5} + \frac{8}{3}i\right)$$

$$= \left(\frac{10}{3} - \frac{26}{5}\right) + i\left(\frac{24}{5} - \frac{8}{3}\right)$$

$$= -\frac{28}{15} + \frac{32}{15}i$$

**IP3:**

Simplify $(1-i)^3(1+i)$.

**Solution:**

$$\begin{aligned}(1-i)^3(1+i) &= (1-i)^2(1-i)(1+i)\\ &= (1+i^2-2i)(1-i^2)\\ &= (1-1-2i)(1+1)\ [\because i^2=-1]\\ &= (-2i)2\\ &= -4i\end{aligned}$$

## IP4:

Find the division: $\frac{5+\sqrt{2}i}{1-\sqrt{2}i}$

**Solution:**

$$\frac{5+\sqrt{2}i}{1-\sqrt{2}i}=\frac{5+\sqrt{2}i}{1-\sqrt{2}i}\times\frac{1+\sqrt{2}i}{1+\sqrt{2}i}=\frac{5+5\sqrt{2}i+\sqrt{2}i-2}{1-\left(\sqrt{2}i\right)^{2}}=\frac{3+6\sqrt{2}i}{1+2}=1+2\sqrt{2}i$$

1. Multiply the complex numbers

$$z_1 = 3 + 2i \text{ and } z_2 = -1 - 4i.$$

2. Represent $\frac{1+3i}{2+i}$ as a complex number.

3. Represent $\frac{3-2i}{2-3i}$ as a complex number.

4. Express $z$ in the form $x + iy$, when $z = \frac{3-2i}{-1+4i}$